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Abstract‡

A cash flow model is developed to set the price for a loan to a borrower with known risks. Similarities are noted between this model and those used for profit testing in life insurance. We emphasize aspects that reasonably can be treated in several ways and also indicate where the cash flow model differs from the pricing methods usually employed in bank lending. The sensitivity of interest rates to various parameters of the model such as the length of loan and the expected default rate is examined. Also, we examine how features of loans, including cash back and early repayments, can be priced.

Key words and phrases: *credit risk, default rate, equity, expenses, mortgages, net present value*

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1 Introduction

The principal objective of a bank is to make loans in such a manner as to provide its shareholders with a healthy return on their equity capital. To this end, banks make large corporate loans, small corporate loans, personal loans (including mortgages and auto loans and unsecured loans) and operate credit cards. For the large loans there is more information required (e.g., financial statements and accounts and the institution's credit rating). Risk pricing, whereby different interest rates are set according to the default risk associated with the loan, is accepted as the normal market practice for setting loan rates. For smaller corporate loans and personal loans there is some credit risk information, both on the economic background and individual risk. The normal market practice for these loans, however, is to charge a uniform price to those who are offered loans. The risk analysis merely determines the decision of whether to lend or not; it does not affect the interest rate charged.

There are several types of risks banks face including:

- Credit risk, i.e., the risk that some borrowers will default; it is a bank's major consideration in the lending process;[1]
- Market risk, i.e., the risk of changes in the market value of assets;
- Liquidity risk, i.e., the risk of not holding enough liquid assets as the bank's liabilities are predominantly short term in nature; and
- Operational risk, i.e., fraud, computer failure, terrorism, etc.

We are concerned primarily with credit risk and its impact on smaller corporate loans and personal loans. The risk of default for these loans is a major consideration because there is often not much relevant information on the borrower's ability to repay the loan. The market, liquidity, and operational risks are discussed by Allan et al., (1998).

Once the potential borrower's credit risk is known,[2] the bank can choose to decline or accept the request for a loan. If the request is acceptable, the bank must decide at what level to set two key parameters:

[1] As banks are aware of the possibility of loan defaults, they make an annual provision for the resulting bad debts, typically 1 percent of outstanding loans. This figure varies as bad debt is sensitive to the state of the economy. Values for the provisions for loans to various industries are given by Davis (1993). More recent ratios are given in the *Banking Act Report* (annual) and the *Annual Abstract of Banking Statistics* (but these do not include industry breakdowns).

[2] A discussion of how credit risk is determined is given in Appendix A.

the interest rate charged on thè loan and the amount of capital set aside to back the loan.

The bank can calculate the minimum interest rate required to provide sufficient returns to the bank and then compare this with the current market rate of interest for this type of loan. Though an increased interest rate will raise the expected proceeds from the loan, it may also increase the borrower's default risk.

The capital allocation is based on two considerations: regulatory and economic. First, there is a regulatory requirement for banks to hold a certain amount of capital to protect the bank from insolvency.[3] In the U.K. banks generally have held capital of around 10 percent of these assets, 6 percent of which has been equity capital.[4] Within each category (e.g., commercial loans or mortgages) the regulatory capital requirement includes no allowance for differences in default risk.[5]

Second, there is a general preference among shareholders for a stable pattern of returns. Variable credit losses can lead to variable returns. This variability of returns can be reduced by holding more capital, as any losses will lead to a smaller percentage loss of capital. The economic capital requirement increases with the variability of credit risk.

Increasing the amount of capital that supports a loan reduces the expected return on capital, however, unless there is an increase in interest rates. Thus the interest rate to charge on the loan and the amount of capital set aside to back the loan are interdependent.

2 An Overview of Basic Cash Flow Models

Cash flow models for bank loans have a variety of uses such as: (i) calculating the return on equity capital to see whether lending is likely to be profitable at a particular interest rate, (ii) examining the impact of various parameters on default scenarios, and (iii) examining the cost to

[3]Banks are required under the Basle Accord to hold capital of at least 8 percent of their risk-weighted assets, including at least 4 percent equity capital; the remainder will be debt capital. Risk-weighted assets include 100 percent of commercial loans, 50 percent of mortgages, and 0 percent of government debt.

[4]The amount of capital held varies with time, e.g., both quantities increased through the first half of the 1990s, and varies between banks, with some banks holding total capital of 14 percent and equity capital of 9 percent.

[5]This means that loans to large corporations need as much capital backing (per £ of loan) as do loans to individuals. This contrasts with risk-adjusted or economic capital that takes risk into account. It is a concern among banks that this encourages high risk lending, as it is inefficient to hold large amounts of capital for low risk loans.

the bank of loan features such as guaranteeing fixed interest rates or allowing early repayments. In considering the makeup of a cash flow model, however, we will focus on calculations of expected net present value and return on equity capital.

Cash flow equations in bank lending may be complicated, but the ideas are not different from those used in other actuarial cash flow models. There are terms for the amount of inflow and outflow, the timing of these flows, the probability that they occur, and a discount factor. The complexity arises because there are many parties to consider (the shareholders, the borrower, the bank's treasury, and the providers of debt capital). In addition there is a possibility of premature termination of the loan by default or early repayment, both of which yield income (including default recoveries and surrender fees).

A cash flow model can be based on the total amount of loans outstanding and other directly linked quantities such as capital, monthly expenses, monthly net interest income, losses due to defaults, and fees from early repayments. These quantities may be fixed or variable.

There are three approaches to examining cash flows relating to lending: the cohort loan approach, steady state portfolio approach, and the whole business approach. These approaches have different uses and they do not give the same value for the profitability of a particular class of business.

Cohort Loan Approach: Only the income and outgo relating to one or a group (cohort) of similar loans issued at the same time are considered.

Steady State Portfolio Approach: Here lending is viewed as a steady state process whereby at any time a given block of loans is outstanding, it is supported by a proportional amount of capital. The outstanding loans give rise to streams of interest payments and expenses. The development of individual loans is ignored for such calculations.[6] For a bank that already has many loans written and expects to both issue new loans and receive final payments on others at a steady rate, it is not necessary to consider each loan in detail. (Although it is probably useful to consider a set of loans from start-up when pricing.)

Whole Business Approach: We consider the whole business of lending including (i) the costs of establishing computer systems, training

[6]A variant is when the loan book is expected to fluctuate, but the entire set of loans still is considered rather than each loan. This is a simpler, more practical approach to the analysis of loan cash flows than studying each loan. It omits some details relating to the timing of payments.

staff, and so on; (ii) a model of the growth rate of the business; and (iii) all of the cash flows arising directly from the lending. This differs from the other approaches by including more expenses (not just those directly related to marketing and maintaining loans).

The cohort loan valuation method can be inappropriate when the arrangements for repaying the funds for the loans and for paying the expenses generated by the loan are based on the portfolio of loans. In this case, it would be possible to use the proportional repayment calculations in the individual loan cash flows to handle the funding costs, but it would still be necessary to make decisions regarding what portion of the net income generated by a particular loan in each month is to be paid to the providers of capital and what portion is to be used to meet expenses of the portfolio.

The steady state method cannot readily be used for pricing new business or considering the profitability of a new type of loan. It is best to use individual loans to assess the value of features such as initial discounts or early repayments, because the timing of payments is crucial in this instance. When looking at the whole portfolio, income generated now is compared with the cost of capital in place now rather than being matched with the capital that was invested in the past to back the loans that are now generating income.

When deciding on the profitability of a new line of business (for example, personal loans sold by telephone), the whole business approach may be better as calculating the value of each loan is not sufficient. There will be substantial start-up costs and marketing expenses may be higher per loan arranged in the first year compared with loans made later. These extra costs must be spread across all loans of this class made over a period of several years. A cash flow analysis must include these initial costs, estimates of the growth in volume of lending (e.g., quarterly estimates for the first five years), and the income and outgo pertaining to each loan.

As we are primarily considering interest rate setting and the profitability of a tranche of loans, we will use the cohort loan approach.[7]

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[7] A note on the words used in this paper: *cohort* and *tranche* are used when describing loans issued at the same time; *steady state*, *portfolio*, and *book* are used to describe a combination of loans at different stages of development. The phrase *set of loans* is used for either of these two situations, i.e., it is an alternative to using the plural *loans*.

3 Cash Flow Model for a Cohort of Loans

The cash flow model is developed sequentially. First we consider only the loan and the equity capital, with expenses, debt capital, defaults, and early repayments being ignored. These items are introduced singly later in the paper.

3.1 Two Sources of Funds

Each cash flow resulting from a loan can be split into two sources: flows that belong to the shareholders and flows that do not belong to the shareholders. To assess the profitability of a loan it is essential to correctly identify from which source each element of a cash flow came. This idea is developed further in the following example, with expenses ignored for simplicity. Let

r_F = Cost of funds, which is at least the money market rate and possibly larger to allow for the expenses of the treasury department;

r_L = Interest received on the loan;

r_C = Interest earned by the bank's equity capital; and

C_t = Net cash flow at t.

To make a one year loan of say, 100, at $r_L = 12$ percent, the bank's lending department will, in turn, have to borrow the same amount of money from the bank's treasury department. The bank's treasury department in turn will acquire the money from retail deposits or short-term borrowing in the wholesale markets. The treasury will charge the lending department a rate $r_F = 10$ percent for the use of this money. It is the two percent difference between r_F and r_L percent that is the crucial element in the profitability calculations.[8]

The bank also must set aside equity capital of 5 percent of the loan to back each loan. These funds will be invested in the money markets and earn $r_C = 8$ percent during the year. Depending on how the bank's treasury operates, r_C could be equal to r_F.

Thus, as far as the shareholders are concerned, the initial cash flow is $C_0 = -5$, i.e., the capital set aside. The end of year cash flow is

[8] In the cohort approach, the global weighted average margin on all loans would be determined so that it was sufficient to provide an appropriate return on capital. There would be insufficient explicit consideration given to the cash flow pertaining to individual loans.

$$C_1 = 100(1 + r_L) - 100(1 + r_F) + 5(1 + r_C) = 112 - 110 + 5.4 = 7.4.$$

Note that these cash flows that belong to the shareholders are small in comparison with the total cash flows that occur in the lending process.

The profitability of this loan is related to the net present value (*NPV*) which is given by

$$NPV(r) = C_0 + \frac{C_1}{1 + r}$$

where r an interest rate. A loan is profitable if $NPV(r_H) > 0$ where r_H is the hurdle rate.[9] The internal rate of return (*IRR*) is the rate of interest that solves $NPV(IRR) = 0$. In most cases where *IRR* is greater than the hurdle rate the project will be sufficiently profitable. In this example, with a (pre-tax) hurdle rate of 20 percent, we have $NPV(0.20) = 1.17$ and an *IRR* of 48 percent. Thus, with no expenses or defaults, it is sufficiently profitable to lend with these rates of interest.

In practice a more complicated method may be used to determine if a loan is sufficiently profitable. This method involves (i) calculating *NPV* at above the hurdle rate, with a check that this is positive; (ii) calculating *IRR*, with a check to see that it is sufficiently high; and (iii) a check on the sensitivity of *NPV* to relevant variables.

The implications of having two sources have been detailed because, although splitting cash flow may be obvious, this situation is not mentioned in standard business finance texts in discussing the valuation of cash flows. One method mentioned in texts is to compare the *IRR* of all the flows with an average of the returns required by those involved with the project (here the providers of capital and the bank's treasury). This example gives a combined initial outgo of 105 and the final income of 117.4, yielding

$$NPV(r) = -105 + \frac{117.4}{1 + r}$$

and $IRR = 11.81$ percent. Some authors have suggested that *NPV* could be calculated using a discount rate equal to a weighted average of the

[9]The hurdle rate is set by the bank according to the riskiness of the loan using a risk versus return model such as the capital asset pricing model. It is higher than the rate of interest charged by the treasury because the treasury is exposed to less risk than the loan department. The treasury has a prior claim on any income; if there is any shortfall (e.g., because of a loan default) capital, if available, will be used to make up the difference.

two rates of interest involved. This is called the *weighted average cost of capital approach.*[10] This method is not as precise as considering the flows to and from each participant separately.

The separation of a cash flow into several streams is familiar to actuaries—for example in the context of unit-linked life policies (e.g., Squires, 1986) where premium income is split between a unit fund (belonging to the policyholder) and a sterling fund (belonging to the office). A closer analogy to the two sources of funds required in bank lending is where a negative sterling fund is used in a life office (e.g., Hare and McCutcheon, 1991). In such a situation the initial strain caused by establishing a policy is partly backed by capital, which requires one rate of interest, and partly by internal funds, which require a lower rate of interest.

3.2 The Basic Mathematical Model

We begin with a basic cash flow model that consists of loan repayments from the borrower to the loan department and from the loan department to the treasury.

In general, most personal loans or mortgages are amortized over time by level installments that include both interest and principal elements. An alternative approach is to use a sinking fund arrangement where a series of interest only payments are made and a final complete repayment of the principal. The sinking fund approach has capital outstanding for a longer period and therefore may have a greater risk to the bank than the amortization approach. In the amortization situation there is also a release of capital each month, as the capital requirement is likely to be proportional to the amount of the loan outstanding.

The amortization method is used throughout this paper. Without loss of generality, we assume loans are repaid on a monthly basis. The following notations are needed:

[10] See, for example, Higson 1986, Chapter 16, or Brealey and Myers, 1991, Chapter 19.

$$\begin{aligned}
L_0 &= \text{Size of loan;}\\
X &= \text{Size of the level monthly installment to amortized } L_0;\\
L_t &= \text{Loan outstanding at end of month } t,\ t = 1, 2, \ldots;\\
K_0 &= \text{Initial capital;}\\
n &= \text{Duration of loan in months;}\\
i_L &= \text{Monthly interest rate on loan;}\\
i_F &= \text{Monthly interest rate on funds;}\\
i_C &= \text{Monthly interest rate on set aside capital;}\\
i_H &= \text{Monthly hurdle rate; and}\\
v_H &= 1/(1 + i_H).
\end{aligned}$$

Note that throughout this paper the symbol r is the annual percentage rate (APR) corresponding to i. So, for example, $(1 + i_L)^{12} = 1 + r_L$.

It is well known that X and L_t are given by:

$$X = \frac{L_0}{a_{\overline{n|}\, i_L}} \tag{1}$$

$$L_t = X a_{\overline{n-t|}\, i_L} \tag{2}$$

where $a_{\overline{n|}\, i}$ is the present value of an annuity of one per month paid in arrears for n months evaluated at interest rate i.[11]

Let B_t denote the amount paid to the bank's treasury at the end of month t. Two possible schemes are considered for B_t, a uniform scheme and a proportional scheme. These schemes lead to

$$B_t = \begin{cases} L_0 / a_{\overline{n|}\, i_F} & \text{Uniform Scheme;} \\ (1 + i_F)L_{t-1} - L_t & \text{Proportional Scheme.} \end{cases} \tag{3}$$

The uniform repayment scheme involves n equal payments to the treasury; this is the same pattern as the initial intended payments by the borrower to the bank. The proportional scheme assumes that, at the start of each month, the bank borrows an amount equal to the loan outstanding at the time and repays this with interest at the cost of funds rate at the end of the month. It implicitly assumes that it will be

[11] As this paper does not focus on risks relating to changes in base rates, these formulae for X and L_t have been based on a constant interest rate throughout the term of the loan.

possible, throughout the term of a long loan for the treasury to be able to borrow the amount of money that already has been lent by the bank. Note that if a loan is repaid early, the uniform repayment plan ignores this while the proportional plan adapts by bringing forward the return of money to the treasury.

Both patterns have advantages: the uniform method fixes in advance the interest paid on the borrowed funds (the margin over base rate is fixed) so that uncertainty about future movements in these interest rates can be removed from the lending decision. The treasury knows in advance the pattern of payments it will receive from the lending department. The proportional method ensures that the amount borrowed at any time is the same as the amount being lent.

For the remainder of this paper we will use the proportional repayment method as this equates more closely to procedures followed in practice.[12]

Some more notation is required for cash flow modeling:

$$\begin{aligned} K_t &= \text{Equity capital outstanding at end of month } t; \\ &= \frac{K_0 a_{\overline{n-t|}\, i_L}}{a_{\overline{n|}\, i_L}}; \qquad (4) \\ RTK_t &= \text{Equity capital returned at the of month } t; \\ &= K_{t-1} - K_t; \qquad (5) \\ i_C K_{t-1} &= \text{Interest earned on equity capital during month } t. \end{aligned}$$

The release of capital implied by these definitions matches the repayments of principal by the borrower; thus the amount of capital is kept in proportion to the loan outstanding. This procedure should not be followed if analysis suggests that the loan is becoming more risky. The capital backing the lending should be kept at a level sufficient to cover future losses.

Using the basic model, the net monthly income (*NMI*) and net present value of the loan, from the viewpoint of the shareholders, is given by

$$NMI_t = X - [(1 + i_F)L_{t-1} - L_t] + i_C K_{t-1} + RTK_t \qquad (6)$$

$$NPV(i_H) = -K_0 + \sum_{t=1}^{n} NMI_t v_H^t. \qquad (7)$$

[12] Appendix B contains a discussion of the differences arising under the uniform repayment pattern and includes an explanation of why what appears to be a bookkeeping choice is more important to the derived profitability of a loan than are real features such as default rates.

Note that this value depends on the decision on how to arrange the repayments to the bank's treasury, the withdrawal of capital each month, and the borrower's repayment pattern.

Example 1

Tables 1 and 2 illustrate the development of the various terms in the cash flow equation using the following parameters: $L_0 = 5,000$, $K_0 = 0.05L_0$, $n = 36$ months, $r_H = 20$ percent, $r_L = 12$ percent, $r_F = 10$ percent, and $r_C = 8$ percent.

Table 1
Cash Flows in Respect of Borrower's Repayments

Month	(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	5000.000	47.444	117.166	164.610	39.871	117.166	157.037
2	4882.834	46.332	118.278	164.610	38.936	118.278	157.215
3	4764.555	45.210	119.401	164.610	37.993	119.401	157.394
4	4645.155	44.077	120.533	164.610	37.041	120.533	157.575
5	4524.621	42.933	121.677	164.610	36.080	121.677	157.757
6	4402.944	41.779	122.832	164.610	35.110	122.832	157.941
7	4280.112	40.613	123.997	164.610	34.130	123.997	158.128
8	4156.115	39.437	125.174	164.610	33.141	125.174	158.315
9	4030.941	38.249	126.362	164.610	32.143	126.362	158.505
10	3904.580	37.050	127.561	164.610	31.136	127.561	158.696
11	3777.019	35.839	128.771	164.610	30.118	128.771	158.890
12	3648.248	34.617	129.993	164.610	29.092	129.993	159.085
13	3518.255	33.384	131.226	164.610	28.055	131.226	159.281
14	3387.029	32.139	132.472	164.610	27.009	132.472	159.480
15	3254.557	30.882	133.729	164.610	25.952	133.729	159.681
20	2573.104	24.416	140.195	164.610	20.518	140.195	160.713
25	1858.701	17.637	146.974	164.610	14.822	146.974	161.795
30	1109.754	10.530	154.080	164.610	8.849	154.080	162.929
35	324.593	3.080	161.530	164.610	2.588	161.530	164.119
36	163.063	1.547	163.063	164.610	1.300	163.063	164.363

Notes: Column (1) = Loan at start of month; Column (2) = Interest paid by borrower; Column (3) = Return of principal by borrower; Column (4) = Total paid by borrower; Column (5) = Interest paid to treasury; Column (6) = Return of principal to treasury; and Column (7) = Total paid to treasury.

Table 2

Cash Flows to Capital

Month	(1)	(2)	(3)	(4)	(5)	(6)
1	250.000	1.609	5.858	15.040	0.985	14.813
2	244.142	1.571	5.914	14.881	0.970	14.435
3	238.228	1.533	5.970	14.719	0.955	14.064
4	232.258	1.494	6.027	14.557	0.941	13.698
5	226.231	1.456	6.084	14.393	0.927	13.340
6	220.147	1.416	6.142	14.227	0.913	12.987
7	214.006	1.377	6.200	14.060	0.899	12.641
8	207.806	1.337	6.259	13.891	0.886	12.301
9	201.547	1.297	6.318	13.720	0.872	11.967
10	195.229	1.256	6.378	13.548	0.859	11.639
11	188.851	1.215	6.439	13.374	0.846	11.316
12	182.412	1.174	6.500	13.199	0.833	10.999
13	175.913	1.132	6.561	13.022	0.821	10.688
14	169.351	1.090	6.624	12.843	0.808	10.382
15	162.728	1.047	6.686	12.663	0.796	10.082
20	128.655	0.828	7.010	11.735	0.738	8.660
25	92.935	0.598	7.349	10.762	0.684	7.361
30	55.488	0.357	7.704	9.742	0.634	6.176
35	16.230	0.104	8.077	8.673	0.588	5.096
36	8.153	0.052	8.153	8.453	0.579	4.892
Sum of Present Value of Monthly Cash Flows						336.50
Less Initial Capital Allocation						-250.00
Net Present Value of Loan						86.50

Notes: Column (1) = Capital at start of month; Column (2) = Interest earned on capital; Column (3) = Return of capital; Column (4) = Net cash flow at end of month; Column (5) = Discount factor; and Column (6) = Net present value.

The net present value is 86.5, the *IRR* is 54.16 percent, and the loan interest rate that would provide a zero *NPV* at a 20 percent hurdle rate is 10.58 percent. If the hurdle rate is 20 percent, 10.6 percent can be regarded as the minimum loan interest rate, ignoring expenses and defaults.

Table 1 shows the constant repayments (164.610) by the borrower split between decreasing interest payments and increasing principal repayments. The total amount paid to the treasury increases each month under the proportional repayment scheme. (Under the uniform repayment scheme the monthly payment to the treasury would be 160.326.) Table 2 shows that each month's net cash flow is positive, and the size decreases as the size of the loan reduces. The net cash flow is calculated using:

$$\begin{aligned}\text{Net cash flow} \quad = \quad & \text{Total paid by borrower} \\ & - \text{Total paid to treasury} \\ & + \text{Interest earned on capital} \\ & + \text{Return of capital.}\end{aligned}$$

3.3 Inclusion of Expenses

There are several ways to deal with expenses, particularly initial expenses, and these methods lead to different values for the profitability of a loan and different sensitivities of the return on equity capital to parameters such as default rate.

Expenses are incurred in establishing the loan, maintaining it, and closing it. The cash flow treatment for these three types of expense is best considered separately.

The initial expenses included in the loan pricing calculations refer only to the costs directly attributable to selling and establishing new loans. They do not include overhead costs or the costs of establishing a line of business. Initial expenses may be met (i) by borrowing from the treasury, (ii) by using equity capital, or (iii) from the net income of existing loans. These three methods are discussed below.

If the initial expenses are borrowed from the treasury they must be repaid, with interest, at some later time using the repayments received on the loan. One way of accounting for this is to amortize these expenses over the term of the loan (using the interest rate applying to the cost of funds); therefore, a portion of each loan repayment will be applied to these start-up costs. This is equivalent to the uniform method of repaying borrowed funds. An alternative is to use the equivalent of the proportional method for the repayment of funds. For example, if the initial expenses are 1 percent of the loaned amount, any future payments to the treasury will be increased 1 percent to allow for the cost of these expenses, including interest. This proportional method is adopted here.

The equations for NMI_t and NPV, including the proportional method of repayment of initial expenses, E_0, are:

$$NMI_t = X - [(1+i_F)L_{t-1} - L_t]\frac{L_0 + E_0}{L_0} + i_C K_{t-1} + RTK_t \quad (8)$$

$$NPV(i_H) = -K_0 + \sum_{t=1}^{n} NMI_t v_H^t. \quad (9)$$

Initial expenses could be met from capital (excluding regulatory capital) on the grounds that there is a risk that they will not be recovered because the borrower fails to make sufficient payments. As default probabilities tend to decrease over the life of the loan, the likelihood of recovering these initial expenses is maximized if the first few installments paid by the borrower are used to meet the expenses rather than contribute to profit. (In life insurance profit testing the initial expenses generally are charged to capital.) But because initial expenses can be relatively large, it would require a substantial increase in the capital outlay for a loan if the expenses had to be met in this way. Moreover, this capital would be consumed immediately and therefore would not earn any interest. Hence, this would be a costly approach. It is also not an approach used in practice.

If the initial expenses are met by capital rather than borrowing, the E_0 term is not needed in equation (8) for net monthly income. The *NMI* and *NPV* equations must be changed to

$$NMI_t = X - [(1+i_F)L_{t-1} - L_t] + i_C K_{t-1} + RTK_t \quad (10)$$

$$NPV(i_H) = -K_0 - E_0 + \sum_{t=1}^{n} NMI_t v_H^t. \quad (11)$$

As E_0 will not, in general, vary directly with L_0, *NPV* will not vary directly in proportion to L_0; thus, small loans will be unprofitable except at high interest rates.

3.4 Debt Capital

As well as holding equity capital, the bank will hold debt capital (also called tier 2 capital) as part of the regulatory requirements. Providers will require a return in excess of what the bank can earn by putting the money in the cash market. This generates an extra expense each month of ${}^{D}K_{t-1} \times (i_D - i_C)$ where ${}^{D}K_t$ is the amount of debt capital held at the end of month t, which will depend on the size of the loan outstanding, and i_D is the monthly interest which has to be paid on this debt.

These payments are included in the cash flow model in the same way as the running expenses, E_t:

$$
\begin{aligned}
NMI_t &= X - [(1 + i_F)L_{t-1} - L_t]\frac{L_0 + E_0}{L_0} + i_C K_{t-1} + RTK_t \\
&\quad - E_t - {}^{D}K_{t-1}(i_D - i_C) \\
NPV(i_H) &= -K_0 + \sum_{t=1}^{n} NMI_t v_H^t.
\end{aligned}
$$

The interest payments relating to debt capital are important but changes in the amount of debt capital held do not alter the cash flows to or from the providers of equity capital.

Example 2

Table 3 and 4 show the behavior of the terms in this equation. The parameters are the same as for Example 1, but with the inclusion of the additional terms: $E_0 = 50$, $E_t = 0$, $r_D = 10$ percent, and ${}^{D}K_0 = K_0$ (i.e., the initial debt capital and equity capital are equal). Here the initial costs are met by borrowing from the treasury, and this produces a net present value of 36.20 (at a discount rate of 20 percent), an internal rate of return of 34.11 percent, and a break-even loan interest rate (at the 20 percent hurdle rate) of 11.41 percent. If capital were used for these initial costs the *NPV* would be 30.28, the *IRR* would be 29.60 percent, and the break-even loan rate would be 11.50 percent.

3.5 Loans Defaults

Some borrowers will default on the repayment of their loans. On some of the defaulted loans, the bank will be unable to recover the full amount of the outstanding principal resulting in a loss.

Table 3
Cash Flows in Respect of Borrower Allowing For Expenses

Month	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1	5000.000	47.444	117.166	164.610	5050.000	40.269	118.338	158.607
2	4882.834	46.332	118.278	164.610	4931.662	39.326	119.461	158.787
3	4764.555	45.210	119.401	164.610	4812.201	38.373	120.595	158.968
4	4645.155	44.077	120.533	164.610	4691.606	37.412	121.739	159.150
5	4524.621	42.933	121.677	164.610	4569.868	36.441	122.894	159.335
6	4402.944	41.779	122.832	164.610	4446.974	35.461	124.060	159.521
7	4280.112	40.613	123.997	164.610	4322.914	34.472	125.237	159.709
8	4156.115	39.437	125.174	164.610	4197.676	33.473	126.426	159.898
9	4030.941	38.249	126.362	164.610	4071.251	32.465	127.625	160.090
10	3904.580	37.050	127.561	164.610	3943.625	31.447	128.836	160.283
11	3777.019	35.839	128.771	164.610	3814.789	30.420	130.059	160.478
12	3648.248	34.617	129.993	164.610	3684.730	29.383	131.293	160.675
13	3518.255	33.384	131.226	164.610	3553.438	28.336	132.539	160.874
14	3387.029	32.139	132.472	164.610	3420.899	27.279	133.796	161.075
15	3254.557	30.882	133.729	164.610	3287.103	26.212	135.066	161.278
20	2573.104	24.416	140.195	164.610	2598.835	20.723	141.597	162.320
25	1858.701	17.637	146.974	164.610	1877.288	14.970	148.443	163.413
30	1109.754	10.530	154.080	164.610	1120.852	8.938	155.621	164.559
35	324.593	3.080	161.530	164.610	327.839	2.614	163.146	165.760
36	163.063	1.547	163.063	164.610	164.694	1.313	164.694	166.007

Notes: Column (1) = Loan at start of month; Column (2) = Interest paid by borrower; Column (3) = Return of principal by borrower; Column (4) = Total paid by borrower; Column (5) = Amount owed to treasury at start; Column (6) = Interest paid to treasury; Column (7) = Return of principal to treasury; and Column (8) = Total paid to treasury.

Table 4
Cash Flows to Capital Allowing For Expenses

Month	(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	250.000	1.609	5.858	0.385	13.085	0.985	12.887
2	244.142	1.571	5.914	0.376	12.932	0.970	12.545
3	238.228	1.533	5.970	0.367	12.779	0.955	12.209
4	232.258	1.494	6.027	0.358	12.623	0.941	11.879
5	226.231	1.456	6.084	0.348	12.467	0.927	11.555
6	220.147	1.416	6.142	0.339	12.308	0.913	11.236
7	214.006	1.377	6.200	0.330	12.149	0.899	10.923
8	207.806	1.337	6.259	0.320	11.988	0.886	10.616
9	201.547	1.297	6.318	0.310	11.825	0.872	10.314
10	195.229	1.256	6.378	0.301	11.661	0.859	10.017
11	188.851	1.215	6.439	0.291	11.495	0.846	9.726
12	182.412	1.174	6.500	0.281	11.327	0.833	9.439
13	175.913	1.132	6.561	0.271	11.158	0.821	9.158
14	169.351	1.090	6.624	0.261	10.988	0.808	8.882
15	162.728	1.047	6.686	0.251	10.816	0.796	8.611
20	128.655	0.828	7.010	0.198	9.930	0.738	7.328
25	92.935	0.598	7.349	0.143	9.001	0.684	6.156
30	55.488	0.357	7.704	0.085	8.027	0.634	5.089
35	16.230	0.104	8.077	0.025	7.006	0.588	4.117
36	8.153	0.052	8.153	0.013	6.796	0.579	3.933
Sum of Present Value of Monthly Cash Flows							286.20
Less Initial Capital Allocation							-250.00
Net Present Value of Loan							36.20

Notes: Column (1) = Capital at start of month; Column (2) = Interest earned on capital; Column (3) = Return of capital; Column (4) = Net interest on debt capital; Column (5) = Net cash flow at end of month; Column (6) = Discount factor; and Column (7) = Net present value.

The entire loss consists, however, of the unrecovered outstanding principal plus any previous missed interest payments plus any extra expenses incurred in the collection of the loan. Hence, it is possible for the entire loss to exceed the outstanding principal. Thus both the frequency of default and the resulting losses are crucial factors in the pricing of loans.

The notation for the cash flow model requires the following additions:

$$
\begin{aligned}
q_t &= \text{Probability of loan default during month } t;\\
P_t^{(q)} &= \text{Probability that loan remains in effect at end of month } t;\\
&= \prod_{j=1}^{t}(1-q_j);\\
f_t &= \text{Expected ratio of the loss during month } t \text{ to } L_t.
\end{aligned}
$$

The q_t and f_t must be estimated in advance, perhaps from historical data relating to similar loans. The estimation of these rates, however, is a major challenge.

The expected loss during month t is $q_t \times P_{t-1}^{(q)} \times f_t \times L_{t-1}$. This formulation of default recovery assumes either that the recoveries are made immediately or that $(1 - f_t)L_{t-1}$ refers to the present value at time t of the amounts recovered at later dates.

The expected net monthly income of the loan, taking account of the defaults, becomes:

$$
\begin{aligned}
NMI_t &= XP_t^{(q)} - [(1+i_F)P_{t-1}^{(q)}L_{t-1} - P_t^{(q)}L_t]\frac{L_0+E_0}{L_0}\\
&\quad + i_C P_{t-1}^{(q)}K_{t-1} + (P_{t-1}^{(q)}K_{t-1} - P_t^{(q)}K_t)\\
&\quad - E_t - P_{t-1}^{(q)}{}^{D}K_{t-1}(i_D - i_C) + q_t P_{t-1}^{(q)}(1-f_t)L_{t-1}. \qquad (12)
\end{aligned}
$$

Even with all of the features that have been incorporated in the cash flow model, the complexity of the situation is understated because loans will not be split between on-going and defaulted. There are likely to be some loans in arrears, for which provisions may be set aside before the default date (or the date of successful repayment of the amount owed). For mortgages a loan can be in arrears for more than two years before the situation is resolved, so this is not just a small matter of detail.

Example 3

This example continues from Example 2 with the inclusion of two new parameters: q_t = 0.2 percent (per month); and f_t = 0.2 (i.e., 80 percent of the outstanding loan is recovered). The columns in Tables 5 and 6 with asterisks have been explicitly adjusted by survival probabilities. (Some of the other columns are sums and thereby acquire the adjustment indirectly.)

The loan is only just profitable at a discount rate of 20 percent with an *NPV* of 1.24. The internal rate of return is 20.47 percent, and the break-even loan interest rate at the 20 percent hurdle rate is 11.98 percent. The total money received from continuing borrowers is less than the amount paid each month to the treasury, and the recovery of a substantial portion of each defaulted loan is an important component of the net monthly income.

Table 7 displays the net present values and internal rates of return using the same parameters as used to construct Tables 5 and 6, but with monthly default rates included. Table 7 demonstrates that the *IRR* is somewhat more variable when the initial expenses are met by borrowing rather than being met from capital.

3.6 Early Repayment of Loans

The terms of a loan sometimes will, for a fee, allow the borrower to repay the loan early. Early repayments can be an important feature of long-term loans such as mortgages where many borrowers move or may switch banks in search of the lowest interest rates. The bank cannot rely on the receipt of a full number of interest payments to provide the required profits. The problem is amplified by the fact that U.K. mortgage loans often include a reduced interest rate in the first year and by the concentration of expenses and default risks near the beginning of the loan period. The bank relies on later interest payments to make lending worthwhile.[13]

The loan is only just profitable at a discount rate of 20 percent with an *NPV* of 1.24. The internal rate of return is 20.47 percent, and the break-even loan interest rate at the 20 percent hurdle rate is 11.98 percent.

[13]In the U.S.A., the problem of early repayment is dealt with mainly through the use of points, i.e., interest paid in advance at start of loan for a reduced interest rate. This reduces the early repayment risk.

Table 5
Cash Flows in Respect of Borrower Allowing For Defaults

Month	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
1	0.998	5000.000	47.349	116.932	164.281	5050.000	40.269	128.201	168.471
2	0.996	4882.834	46.147	117.806	163.953	4931.662	39.247	128.827	168.074
3	0.994	4764.555	44.939	118.686	163.625	4812.201	38.220	129.458	167.678
4	0.992	4645.155	43.725	119.572	163.297	4691.606	37.188	130.095	167.282
5	0.990	4524.621	42.506	120.465	162.971	4569.868	36.150	130.737	166.887
6	0.988	4402.944	41.280	121.365	162.645	4446.974	35.108	131.384	166.492
7	0.986	4280.112	40.048	122.272	162.320	4322.914	34.060	132.037	166.097
8	0.984	4156.115	38.810	123.185	161.995	4197.676	33.007	132.695	165.702
9	0.982	4030.941	37.566	124.105	161.671	4071.251	31.949	133.359	165.308
10	0.980	3904.580	36.315	125.032	161.348	3943.625	30.885	134.029	164.914
11	0.978	3777.019	35.059	125.966	161.025	3814.789	29.817	134.704	164.521
12	0.976	3648.248	33.796	126.907	160.703	3684.730	28.743	135.385	164.128
13	0.974	3518.255	32.526	127.855	160.381	3553.438	27.663	136.072	163.735
14	0.972	3387.029	31.251	128.810	160.061	3420.899	26.578	136.764	163.342
15	0.970	3254.557	29.968	129.772	159.741	3287.103	25.487	137.463	162.950
20	0.961	2573.104	23.457	134.692	158.150	2598.835	19.950	141.043	160.993
25	0.951	1858.701	16.776	139.799	156.574	1877.288	14.267	144.775	159.042
30	0.942	1109.754	9.916	145.099	155.015	1120.852	8.434	148.665	157.098
35	0.932	324.593	2.872	150.599	153.471	327.839	2.442	152.718	155.160
36	0.930	163.063	1.440	151.724	153.164	164.694	1.224	153.549	154.773

Notes: Column (1) = Probability of payment at end of the month; Column (2) = Loan at start of month; Column (3) = Interest paid by borrower; Column (4) = Return of principal by borrower; Column (5) = Total paid by borrower; Column (6) = Amount owed to treasury at start; Column (7) = Interest paid to treasury; Column (8) = Return of principal to treasury; and Column (9) = Total paid to treasury.

Table 6

Cash Flows to Capital Allowing For Defaults

Month	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1	250.000	1.609	6.347	0.385	8.000	11.380	0.985	11.209
2	244.142	1.568	6.378	0.375	7.797	11.245	0.970	10.909
3	238.228	1.527	6.409	0.365	7.593	11.109	0.955	10.614
4	232.258	1.485	6.440	0.356	7.388	10.973	0.941	10.326
5	226.231	1.444	6.472	0.346	7.182	10.836	0.927	10.043
6	220.147	1.402	6.504	0.336	6.975	10.699	0.913	9.766
7	214.006	1.360	6.536	0.326	6.766	10.560	0.899	9.495
8	207.806	1.318	6.569	0.316	6.557	10.422	0.886	9.229
9	201.547	1.276	6.602	0.305	6.347	10.282	0.872	8.968
10	195.229	1.234	6.635	0.295	6.136	10.142	0.859	8.713
11	188.851	1.191	6.669	0.285	5.923	10.002	0.846	8.462
12	182.412	1.148	6.702	0.275	5.710	9.861	0.833	8.217
13	175.913	1.105	6.736	0.264	5.496	9.719	0.821	7.977
14	169.351	1.062	6.771	0.254	5.280	9.577	0.808	7.742
15	162.728	1.018	6.805	0.244	5.063	9.433	0.796	7.511
20	128.655	0.797	6.982	0.191	3.963	8.709	0.738	6.427
25	92.935	0.570	7.167	0.136	2.834	7.967	0.684	5.449
30	55.488	0.337	7.360	0.081	1.675	7.208	0.634	4.569
35	16.230	0.098	7.560	0.023	0.485	6.430	0.588	3.778
36	8.153	0.049	7.601	0.012	0.243	6.273	0.579	3.630
Sum of Present Value of Monthly Cash Flows								251.24
Less Initial Capital Allocation								-250.00
Net Present Value of Loan								1.24

Notes: Column (1) = Capital at start of month; Column (2) = Interest earned on capital; Column (3) = Return of capital; Column (4) = Net interest on debt capital; Column (5) = Recovery from defaulted loans; Column (6) = Net cash flow at end of month; Column (7) = Discount factor; and Column (8) = Net present value.

Table 7
Impact of Expenses on *NPV* and *IRR*
With $E_0 = 50$, $L_0 = 5,000$, and $K_0 = 250$

Monthly Default Rate	Expenses Borrowed From Treasury		Expenses Paid From Capital	
	NPV	*IRR*	*NPV*	*IRR*
0.0%	36.20	34.11%	30.28	29.60%
0.2%	1.24	20.47%	-4.56	18.58%
0.4%	-32.26	8.06%	-37.94	8.40%
0.6%	-64.37	-3.21%	-69.92	-1.00%

The early termination of a loan can be put in a cash flow model in a similar manner to defaults. Let G_t denote the fee charged for early repayment in month t; and R_t denote the probability that a loan that has survived to the end of month t is repaid at that time. The survival probability for a loan becomes

$$P_t^{(qr)} = \prod_{j=1}^{t}(1-q_j)(1-R_j). \tag{13}$$

The proportion of the original loans that default at the end of month t will be $q_t P_{t-1}^{(qr)}$, while the repayments will be $R_t(1-q_t)P_{t-1}^{(qr)}$. The expression for net monthly income becomes:

$$\begin{aligned} NMI_t \quad = \quad & P_{t-1}^{(qr)}X(1-q_t) - [(1+r_F)P_{t-1}^{(qr)}L_{t-1} - P_t^{(qr)}L_t]\frac{L_0+E_0}{L_0} - E_t \\ & + i_C P_{t-1}^{(qr)}K_{t-1} + (P_{t-1}^{(qr)}K_{t-1} - P_t^{(qr)}K_t) \\ & - (i_D - i_C)^D K_{t-1}P_{t-1}^{(qr)} + q_t P_{t-1}^{(qr)}(1-f_t)L_{t-1} \\ & + R_t(1-q_t)P_{t-1}^{(qr)}(L_t+G_t). \end{aligned} \tag{14}$$

The approach we have taken here is an interesting contrast to the approach taken in Allan et al., (1998). That paper assumes that all loans survive the average period of seven years for a U.K. mortgage and then are repaid. Pricing is set so that the average loan provides an appropriate profit. In the U.K. this method would provide reasonable results and would provide similar results to this system where a distribution of future repayment times is used. If there were less inertia in the loan market, a more active fee structure may need to be developed to penalize early repayers or a probability distribution of repayment times may need to be used to estimate the expected cost and variability of cost of early repayment.

Example 4

This example builds on Tables 1 through 6 with the inclusion of early repayments. The two new parameter values are $G_t = 0.01L_t$ and

$$R_t = \begin{cases} 0 & t \leq 12 \\ 0.002 & t > 12. \end{cases}$$

There are four extra columns in Tables 8 and 9: the probability of a loan surviving to the start of the month (i.e., $P_{t-1}^{(qr)}$), the probability of early repayment, the amount of early repayments, and the fees accompanying these repayments.

Because there are no early repayments in the first year of this example, the first twelve months are identical to Example 3. Thereafter, *NMI* is initially greater than in the no repayment example, but in the last months of the loan it is less than in Example 3. *NPV* of 1.52 is marginally higher than without repayments, indicating that the 1 percent fee is sufficient to cover the loss of later positive cash flows. Other values for this loan include an internal rate of return of 20.58 percent and a break-even interest rate of 11.97 percent.

Table 8
Cash Flows in Respect of Borrower Allowing For Prepayments

Month	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
1	1.000	0.998	5000.0	47.3	116.9	164.3	5050.0	40.3	128.2	168.5
2	0.998	0.996	4882.8	46.1	117.8	164.0	4931.7	39.2	128.8	168.1
3	0.996	0.994	4764.6	44.9	118.7	163.6	4812.2	38.2	129.5	167.7
4	0.994	0.992	4645.2	43.7	119.6	163.3	4691.6	37.2	130.1	167.3
5	0.992	0.990	4524.6	42.5	120.5	163.0	4569.9	36.2	130.7	166.9
6	0.990	0.988	4402.9	41.3	121.4	162.6	4447.0	35.1	131.4	166.5
7	0.988	0.986	4280.1	40.0	122.3	162.3	4322.9	34.1	132.0	166.1
8	0.986	0.984	4156.1	38.8	123.2	162.0	4197.7	33.0	132.7	165.7
9	0.984	0.982	4030.9	37.6	124.1	161.7	4071.3	31.9	133.4	165.3
10	0.982	0.980	3904.6	36.3	125.0	161.3	3943.6	30.9	134.0	164.9
15	0.968	0.967	3254.6	29.8	129.3	159.1	3287.1	25.4	143.0	168.4
20	0.949	0.947	2573.1	23.1	132.8	155.9	2598.8	19.7	143.7	163.4
25	0.930	0.929	1858.7	16.4	136.5	152.9	1877.3	13.9	144.5	158.5
30	0.912	0.910	1109.8	9.6	140.2	149.8	1120.9	8.2	145.4	153.6
35	0.894	0.892	324.6	2.7	144.1	146.9	327.8	2.3	146.4	148.8
36	0.890	0.889	163.1	1.4	144.9	146.3	164.7	1.2	146.6	147.8

Notes: Column (1) = Probability of loan surviving to start of month; Column (2) = Probability of payment at end of the month; Column (3) = Loan at start of month; Column (4) = Interest paid by borrower; Column (5) = Return of principal by borrower; Column (6) = Total paid by borrower; Column (7) = Amount owed to treasury at start; Column (8) = Interest paid to treasury; Column (9) = Return of principal to treasury; and Column (10) = Total paid to treasury.

Table 9
Cash Flows to Capital Allowing For Prepayments

Month	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
1	250.0	1.6	6.3	0.4	8.0	0.000	0.000	0.000	11.380	0.985	11.209
2	244.1	1.6	6.4	0.4	7.8	0.000	0.000	0.000	11.245	0.970	10.909
3	238.2	1.5	6.4	0.4	7.6	0.000	0.000	0.000	11.109	0.955	10.614
4	232.3	1.5	6.4	0.4	7.4	0.000	0.000	0.000	10.973	0.941	10.326
5	226.2	1.4	6.5	0.3	7.2	0.000	0.000	0.000	10.836	0.927	10.043
6	220.1	1.4	6.5	0.3	7.0	0.000	0.000	0.000	10.699	0.913	9.766
7	214.0	1.4	6.5	0.3	6.8	0.000	0.000	0.000	10.560	0.899	9.495
8	207.8	1.3	6.6	0.3	6.6	0.000	0.000	0.000	10.422	0.886	9.229
9	201.5	1.3	6.6	0.3	6.3	0.000	0.000	0.000	10.282	0.872	8.968
10	195.2	1.2	6.6	0.3	6.1	0.000	0.000	0.000	10.142	0.859	8.713
15	162.7	1.0	7.1	0.2	5.0	0.002	6.033	0.060	9.697	0.796	7.721
20	128.7	0.8	7.1	0.2	3.9	0.002	4.610	0.046	8.818	0.738	6.507
25	92.9	0.6	7.2	0.1	2.8	0.002	3.179	0.032	7.937	0.684	5.428
30	55.5	0.3	7.2	0.1	1.6	0.002	1.740	0.017	7.054	0.634	4.472
35	16.2	0.1	7.2	0.0	0.5	0.002	0.291	0.003	6.168	0.588	3.624
36	8.2	0.0	7.3	0.0	0.2	0.002	0.000	0.000	5.990	0.579	3.467
Sum of Present Value of Monthly Cash Flows											251.24
Less Initial Capital Allocation											-250.00
Net Present Value of Loan											1.24

Notes: Column (1) = Capital at start of month; Column (2) = Interest earned on capital; Column (3) = Return of capital; Column (4) = Net interest on debt capital; Column (5) = Recovery from defaulted loans; Column (6) = Probability of early repayment; Column (7) = Early repayments; Column (8) = Early repayment fees; Column (9) = Net cash flow at end of month; Column (10) = Discount factor; and Column (11) = Net present value.

3.7 Parameter Dependence

Equation (14) is used to generate Table 10, which shows how the net present value (at a 20 percent hurdle rate), the internal rate of return, and the break-even loan rate (also at a 20 percent hurdle rate) change as the various inputs of the cash flow model are altered. These three values are given for the case where initial expenses are met by borrowing and the case where capital is used for these expenses. The standard model has the following parameters: $L_0 = 5,000$, $K_0/L_0 = 0.05$, $n = 36$ months, $r_L = 12$ percent, $r_F = 10$ percent, $r_C = 8$ percent, $r_H = 20$ percent, $E_0 = 50$, ${}^DK_0/L_0 = 0.05$, $r_D = 10$ percent, $E_t = 0$, $q_t = 0.2$ percent, $f_t = 0.2$, $G_t/L_t = 1$ percent, and $R_t = 0$ for $t \leq 12$ and $= 0.2$ percent for $t > 12$. All other entries in Table 10 differ only in one value. This standard model is the one used in Example 4.

The following observations may be drawn from Table 10:

- In all cases, *NPV* is greater when initial expenses are paid by borrowing rather than by using capital. (The same discount rate has been used for the two scenarios though it may be reasonable to use a lower rate when capital is used.) Clearly the option of borrowing from the treasury is cheaper than using capital to finance expenses. The borrowing option, however, would lead to greater variability of returns on a smaller amount of capital.

- In terms of the increase in break-even interest rate, the effect of the choice between these two methods of paying for the initial expenses is sensitive to the size of the loan, the hurdle rate used, and the amount of initial expenses.

- The loan rate and the cost of funds are more important than the interest rate earned on set aside capital and the interest paid on debt capital. The lending margin between the loan rate and cost of funds dwarfs all other cash flows to shareholders. Therefore the profitability is likely to be heavily dependent on the interest margin.

- The profitability is less sensitive to the hurdle rate than it is to the loan interest rate or the cost of funds. (The loan rate and the cost of fund rates are varied independently in the above table, hence the margin on the loan is changed.)

Table 10
The Impact of Changes in Various Inputs of the Cash Flow Model

Parameter Changes			Borrowed			Capital		
	Old	New	*NPV*	*IRR*(%)	r_L(%)	*NPV*	*IRR*(%)	r_L(%)
Standard			1.52	20.58	11.97	-4.24	18.67	12.07
L_0	5,000	1,000	-35.08	-54.74	14.97	-40.85	-17.34	15.46
	5,000	3000	-16.78	9.33	12.47	-22.54	9.51	12.64
	5,000	10000	47.28	29.05	11.60	41.52	27.16	11.65
K_0/L_0	0.05	0.01	29.42	105.25	11.50	23.66	46.54	11.60
	0.05	0.25	18.96	36.08	11.68	13.20	27.54	11.78
	0.05	0.10	-33.35	13.96	12.57	-39.11	13.58	12.66
n	36	18	-21.66	5.89	12.66	-24.86	6.60	12.76
	36	60	26.97	27.04	11.69	18.40	23.93	11.79
r_L	0.12	0.11	-57.59	-0.37	11.97	-63.34	1.39	12.07
	0.12	0.13	60.59	44.98	11.97	54.80	38.28	12.07
r_F	0.10	0.08	123.75	75.92	9.92	116.78	62.01	10.04
	0.10	0.11	-58.83	-0.70	13.00	-63.99	1.27	13.09
r_C	0.08	0.07	-4.60	18.26	12.08	-10.37	16.78	12.18
	0.08	0.09	7.60	22.93	11.87	1.84	20.58	11.97
r_H	0.20	0.15	15.53	20.58	11.75	12.46	18.67	11.80
	0.20	0.25	-10.94	20.58	12.19	-19.09	18.67	12.34
	0.20	0.30	-22.08	20.58	12.40	-32.37	18.67	12.59

Notes: The original parameter values are given at the start of Section 3.7 and are displayed in the column labeled "OLD" for convenience. The parameter changes are from the column labeled "OLD" to the one labeled "NEW".

Table 10 (Cont.)
The Impact of Changes in Various Inputs of the Cash Flow Model

Parameter Changes			Borrowed			Capital		
	Old	New	*NPV*	*IRR*(%)	r_L(%)	*NPV*	*IRR*(%)	r_L(%)
Standard			1.52	20.58	11.97	-4.24	18.67	12.07
E_0	50	0	45.76	37.58	11.22	45.76	37.58	11.22
	50	100	-42.71	3.70	12.72	-54.24	5.63	12.92
${}^{D}K_0/L_0$	0.05	0.025	4.55	21.74	11.92	-1.21	19.62	12.02
	0.05	0.10	-4.53	18.29	12.08	-10.29	16.80	12.17
r_D	0.10	0.09	4.54	21.74	11.92	-1.23	19.61	12.02
	0.10	0.11	-1.46	19.44	12.02	-7.23	17.74	12.12
E_t	0	1	-25.99	10.66	12.44	-31.76	10.05	12.54
q_t	0.002	0.000	36.30	34.22	11.40	30.41	29.69	11.50
	0.002	0.004	-31.80	8.17	12.55	-37.44	8.49	12.65
f_t	0.2	0.1	17.24	26.72	11.71	11.48	23.66	11.81
	0.2	0.5	-45.62	3.66	12.77	-51.39	4.75	12.87
	0.2	1.0	-124.20	20.09	14.12	-129.96	-15.35	14.22
G_t/L_t	0.01	0.00	0.94	20.36	11.98	-4.83	18.49	12.08
	0.01	0.02	2.11	20.80	11.96	-3.65	18.86	12.06
R_t, $t > 12$	0.2	0.0	1.24	20.47	11.98	-4.56	18.58	12.08
	0.2	0.4	1.80	20.69	11.97	-3.93	18.76	12.07

Notes: The original parameter values are given at the start of Section 3.7 and are displayed in the column labeled "OLD" for convenience. The parameter changes are from the column labeled "OLD" to the one labeled "NEW".

- Because expenses are a higher proportion of small loans, larger loans are more profitable than small loans, both in absolute terms and per unit of capital deployed, all other things being equal. This suggests that differential interest rates with loan size and/or loan fees would be an appropriate charging policy.
- Similarly, long loans produce more profit than short loans because there is a longer time over which to amortize initial expenses.
- The amount of equity capital is more significant than the amount of debt capital because equity capital requires a higher return.
- Both initial expenses and running expenses are important.
- An extra expense of 1 per month on a loan of 5,000 requires the interest rate to be raised 0.5 percent.
- Doubling initial expenses (to 100 per loan of 5,000) would cause a greater loss than doubling the default rates (suggesting that there is a limit to the expense that should be used to assess the default risk of the borrowers).
- Though the default rate is relevant to profitability, the effect of doubling the default rate is no worse than halving the duration of the loan, starting from the parameters of the standard loan.
- Halving the loan loss fraction has a similar impact to halving the loan default rate. This is not surprising, as both parameters relate to the expected loss from a loan.
- With the parameters explored here, early repayments and associated fees are not important.

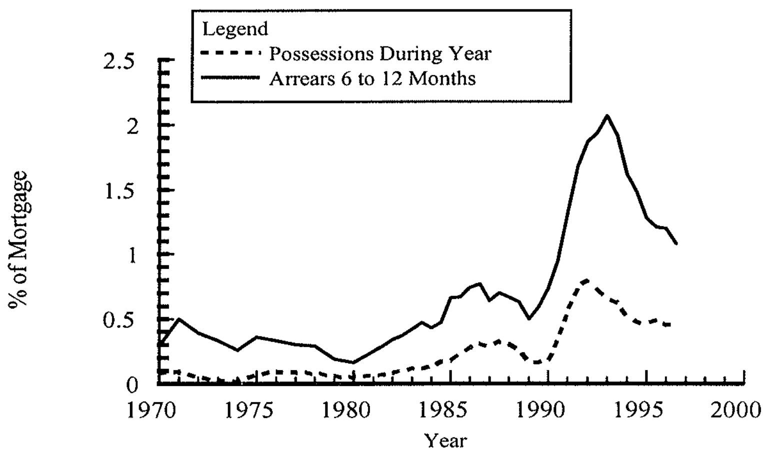
4 Variability of Default Rates and Costs

Data on mortgage arrears and possessions have been collected since 1969 by the Building Societies Association (published in the *BSA Bulletin*) and later the Council of Mortgage Lenders (published in *Housing Finance*).[14] The data suggest evidence of cyclical behavior in the 1970s in the proportions of mortgages ending in possession; see Figure 1. The

[14] A mortgage is said to be in *arrears* whenever at least one scheduled monthly payment is not paid by a certain date. A *mortgage possession* (also called a *repossession*) occurs when the mortgage is in arrears and the bank thus terminates the mortgage and takes ownership of the house. This usually requires a court order.

proportions rose throughout the first half of the 1980s, peaking in the first half of 1987 with an annualized rate of 0.33 percent of mortgaged properties taken into possession. Between the first half of 1989 and the second half of 1991 the annualized rate rose from 0.17 percent of properties repossessed to 0.8 percent. Though there has been a substantial fall since then, one can still assume that future mortgage failure rates will fluctuate considerably over time.

Figure 1
Building Society Possessions and Arrears (*Source: BSA & CML*)



Also, the cost to a bank of defaults on mortgage repayments varies according to the value of the property on which the mortgage is secured. This in turn depends on the change in housing prices since the mortgage was established. In the U.K. the number of mortgage failures was highest at the same time that the cost to the banks was highest, due to falling housing prices. Theoretically, the cost of default to the bank is a compound distribution formed of the probability distribution of defaults and the probability distribution of housing prices (or, more accurately, the difference between the mortgage plus arrears and the value of the house on forced sale). Suitable econometric models of either of these quantities have not been developed for the U.K.; we therefore use the empirical distribution for the cost of default from past data to estimate the sensitivity of the internal rate of return.

To examine the impact of changes in mortgage default rates and housing price inflation we calculate the internal rate of return from mortgage lending using historical data for default rates and house values. The loan model is the same as in Section 3 except that we ignore early repayments. The default rate (q_t) and the loan loss fraction (f_t) are determined from data. Specifically, the loan loss fraction is set by

$$f_t = \max\{0, 0.05 + \frac{H_t - L_t}{L_t}\} \tag{15}$$

where H_t is the housing price at time t. We use a national index of housing price inflation to determine H_t/H_0. The initial housing price is related to the initial size of the loan via the loan-to-value ratio. We consider the extreme case where the initial loan-to-value ratios for mortgages that end in possession are all 100 percent. This maximizes the loss (f_t) and the impact of mortgage defaults on the banks' profitability. The quantity 0.05 in equation (15) represents accumulated arrears and any markdown that occurs when a possessed property is sold.

Data are available on the probability of mortgage failure in a particular year. What are needed for our calculations, however, are conditional probabilities. For example, the probability that a mortgage issued in 1985 failed in 1990, the probability that a mortgage issued in 1986 failed in 1990, and so on.

Assumptions are needed to enable us to estimate the relevant probabilities. We introduce two functions, $\phi(x, y)$ and $Q(x, y)$, which are defined by

$$\begin{aligned}\phi(x,y) &= \text{Pr[Mortgage fails during month } y \mid \text{Mortgage started in} \\ &\qquad \text{month } x \text{ and survived to the start of month } y] \\ &= q_y \bar{N} Q(x,y) \end{aligned} \tag{16}$$

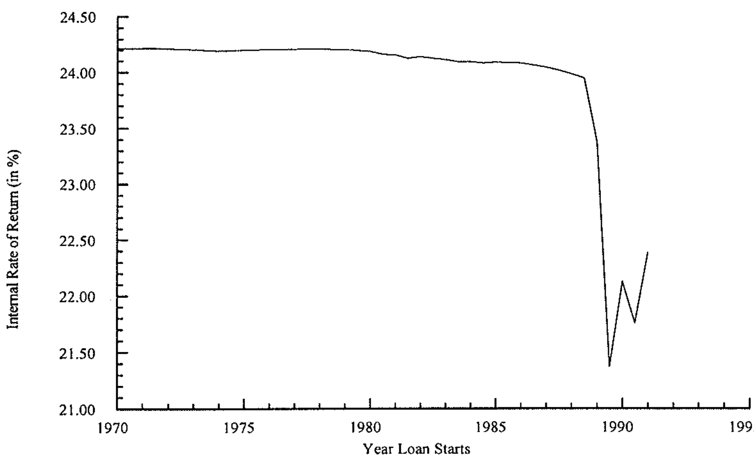
where q_y is the British national default rate (regardless of month of mortgage origin) in month y, according to British data; and $\bar{N}$ is the average length of a mortgage (taken to be 84 months); and $Q(x, y)$ is

$$Q(x,y) = \begin{cases} (y-x)/1200 & y-x = 1, 2, \ldots, 24 \\ 25/1200 & y-x = 25, 26, \ldots, 48 \\ (73-(y-x))/1200 & y-x = 49, 50, \ldots, 72 \\ 0 & y-x = 73, 74, \ldots . \end{cases} \tag{17}$$

The *NPV* for a loan is calculated using equation (12) but with the default probabilities (q_t) replaced by $\phi(x, y)$ of equation (16) and the loan loss fraction given by equation (15).

Figure 2 shows the internal rate of return calculated by the cash flow model using data for default rates and housing price inflation. *IRR* is calculated for successive cohorts of loans. As we assume that no defaults happen more than six years after a loan is made, housing prices and default rates beyond 1997 will have no effect on the profitability of loans issued in 1991 or earlier. The results are calculated using a lending rate of interest r_L = 10.5 percent.

Figure 2
Variability of Returns Due to Changes in Default Rates and House Price Inflation



IRR is insensitive to default rates prior to 1989. Average housing prices peaked in the third quarter of 1989 and fell 12 percent over the next four years. The number of possessions peaked in the second half of 1991. Even in this severe time for the housing market the internal rate of return, based on the assumptions in our model, would have fallen only 3 percent. This illustrates the relatively low risk of mortgage lending due to loans being secured by the value of the borrowers' houses.

The model developed is flexible; for an unsecured loan the compound distribution for the cost of default will depend on default rates (which could be similar to those for mortgages) and the fraction of the loan recovered (which could be less than for mortgages). The variability of *IRR* probably would be much greater. Banks should take into account this risk of default both when setting interest rates that compensate the bank for default and when setting the hurdle rate of return, which should depend on the variability of returns. The hurdle rate should be higher for riskier (unsecured) loans.

5 The Pricing of Features

In this section the net present value will be calculated as a function of interest rate for loans with a variety of features. The purpose is to show the use of cash flow models in assessing how expensive such features are. The loans considered here are not identical to the ones used in the previous sections.

5.1 Cash Back

Many loans are provided that either give the customer some extra cash at the outset of the loan or offer some discount on the loan rate charged for the first year. These features are designed to attract customers to the bank. An alternative would be to offer a constant rate of interest throughout the loan that would be lower than the rate charged in the cash back scheme and lower than the rate charged beyond the first year in the discount scheme.

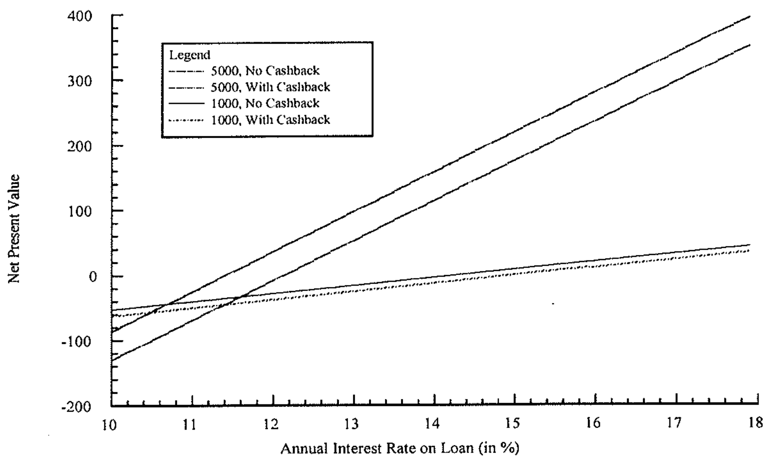
Cash back is included in the cash flow models by treating it as an extra initial expense. For example, if the money needed to provide the cash back is borrowed from the bank's treasury, it may be included in equation (14) by replacing E_0 with $E_0 + CB$ where CB is the amount of cash back. If cash back is paid out of capital, it may be included in equation (11) by replacing E_0 with $E_0 + CB$.

Figure 3 shows the effect that cash back of 1 percent of the loaned amount has on the net present value. The underlying parameters of these loans are the same as used in Example 2. (There are no early repayments or defaults in these calculations). The annual interest rate required to achieve a given NPV is higher by 0.72 percent for both loan sizes; i.e., this is the cost of the cash back. This is a substantial difference in a competitive loan market.

Figure 3 also illustrates a couple of other points: (i) the importance of loan size on the interest rate required to make lending sufficiently profitable (2.9 percent extra for the smaller loan here), and (ii) the greater sensitivity of the profit to interest rates for the larger loan. The slopes of the lines are roughly proportional to the loan size.

Figure 3 is produced using the proportional repayment method for funds and assuming that the initial expenses are paid for by borrowing

Figure 3
Effect of Providing Cashback of 1% of Loan



from the treasury. If initial expenses and cash back are both paid from capital rather than borrowing and the hurdle rate is unchanged, each *NPV* line in Figure 3 will shift to the right (i.e., a higher loan rate is needed to produce a given *NPV*). Small loans are affected more strongly than large loans because the initial expenses are proportionately larger, and likewise the loans with cash back have a larger increase in break-even loan rate than those without cash back (again, because setting aside the capital is expensive).

Table 11 shows the parameters that interact with cash back and those that do not. It also shows how much the break-even loan rate increases if cash back of 1 percent or 5 percent is provided. The values are given for two methods of paying for initial expenses (and also cash back, as this is treated as an additional initial expense), i.e., borrowing or using capital.

These values show that the amount of cash back is important to the break-even loan rate, with the change in this rate being five times greater for the 5 percent cash back loans than for the 1 percent cash back loans. The method of financing the cash back is also important, with the capital payments requiring a larger increase in loan interest rate for a given level of cash back than if the money is borrowed from the treasury.

Table 11
Change in Break-Even Loan Rates, Δr_L, in Percentage Points For Various Levels of Cash Back (CB)

Parameter Changed	New Value	Borrowing 1% CB	Borrowing 5% CB	Capital 1% CB	Capital 5% CB
Standard		0.75	3.75	0.85	4.24
L_0	3,000	0.75	3.75	0.84	4.24
	10,000	0.75	3.74	0.85	4.24
K_0/L_0	2.5%	0.75	3.76	0.84	4.23
	10%	0.75	3.75	0.85	4.26
n	18	1.44	7.27	1.54	7.79
	60	0.47	2.35	0.57	2.84
r_H	15%	0.76	3.81	0.81	4.06
	25%	0.74	3.69	0.88	4.42
E_0	0	0.74	3.74	0.84	4.23
	100	0.75	3.75	0.85	4.24
E_t monthly	1	0.75	3.75	0.85	4.24
q_t	0%	0.73	3.64	0.82	4.13
	0.4%	0.77	3.86	0.87	4.36
f_t	0.1	0.75	3.74	0.84	4.23
	0.5	0.76	3.76	0.85	4.26
	1	0.76	3.78	0.85	4.28
G_t	0%	0.75	3.75	0.85	4.24
	2%	0.75	3.75	0.85	4.24
R_t $(t > 12)$	0%	0.74	3.72	0.84	4.21
	0.4%	0.75	3.77	0.85	4.26

Of the ten parameters varied in Table 11 (and for the range of values examined), seven have negligible interactions with the cost of cash back: the size of the loan, the capital backing for the loan, the initial expenses, the running expenses, the loan recovery fraction, early repayment fees, and early repayment rates (but see the comment on default rates below).

By far the most important factor in terms of the cost of cash back (other than the amount of cash back) is the length of the loan. A short loan requires the same cost to be met by fewer monthly payments; hence, a greater interest rate margin is needed. Likewise, a lower margin is needed for longer loans.

The two other parameters that have some (albeit minor) impact on the cost of cash back are the loan default rate and the hurdle rate of interest. The default rate's influence is due to the alteration of the average duration of the loan. (The early repayment rate is less important because the model excludes any repayments in the first year, so the impact on average duration of a change in this rate is smaller than that of the default rate.) The influence of the hurdle rate is more important for the loans where the initial expenses and cash back are paid using capital. A low hurdle rate makes the initial capital outlay on cash back less expensive in terms of the size of future positive cash flows demanded and does not require such large margins to be paid by the borrower. The cost of cash back is increased at low hurdle rates when the money for it is borrowed (this cost is decreased if capital is used).

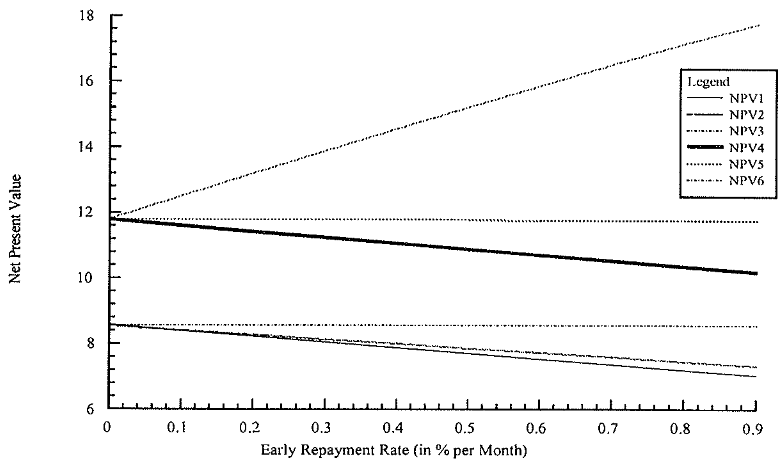
5.2 Early Repayment and Fees

For the loans examined, early repayments are not a significant factor in terms of loan pricing. For example, using the parameters in Example 3, which has no early repayments, the break-even loan rate is 11.98 percent. If early repayments happen at the high rate of 3 percent per month for the second and third years of the loan (in which case more than half of the loans are repaid early) and no early repayment fee is charged, the break-even interest rate only rises to 12.05 percent. Early repayments are more significant if the ratio of initial expenses to the size of the loan is high and the loan is short. Nevertheless, a bank may prefer to set the interest rate appropriate to the full term of the loan and charge early repayers a fee to compensate for missed future interest payments. We find in Allan et al., (1998) that early repayments are a greater problem if higher expenses are assumed at the outset.

The fees necessary to maintain *NPV* in the event of early repayment of loans have been calculated for two loan sizes (1,000 and 5,000) where the interest rates charged on the loans are 15 percent and 11.6 percent, respectively, and other parameters are the same as for previous examples except that there are no defaults and no cash back. Initial expenses are financed by borrowing from the treasury. There are no repayments in the first 12 months, and the rate is constant thereafter.

Results are shown in Figure 4 where *NPV* is plotted against the early repayment rate (R_t). Two fans of three lines are shown; the upper is for the $L_0 = 5,000$, $r_L = 11.6$ percent model, the lower is for the other pair of values. In each case the lowest line of the three lines is *NPV* if no fee is charged for early repayment. (The proportional fund repayment scheme has been used here so that the lines slope downward as early repayments increase.) Let G_t denote the fee actually charged for early repayment at time t. The middle line in Figure 4 reflects a fee of 0.6 percent of the loan outstanding, i.e., $G_t = 0.006L_t$, and the upper line reflects a fee of 2.9 percent of the loan outstanding, i.e., $G_t = 0.029L_t$.

Figure 4
Effect of Early Repayment and Fees

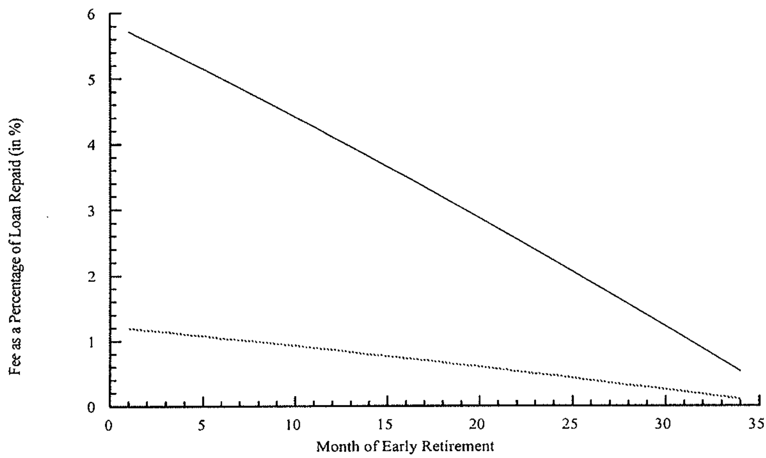


For the smaller loan the repayment fee needs to be at least 2.9 percent of the outstanding loan if the bank is not to lose by allowing repayments. But for the larger loan a proportionally smaller fee (i.e., about 0.6 percent) is needed. Thus, loan size is relevant to the impact of early repayment fees. The fee expressed as a percentage of the loan is inversely proportional to the loan size (although, the ratio 2.9 percent to 0.6 percent is close to the inverse of the ratio of loan sizes, i.e., 1,000 to 5,000).

Figure 4 shows that there is a linear relation between early repayment rate and net present value. It is possible to find a single fee (for a given set of loan parameters) that makes the bank broadly neutral to the frequency of early repayments.

The cost of an early repayment depends on the time that it happens, with the earlier repayments being more of a problem than those that happen close to the full term of the loan. Figure 5 shows the early repayment fee necessary to keep NPV of a loan constant; this fee changes with time and is calculated using a prospective approach. The fee is equated to the present value, at the time of early repayment, of the future net monthly income that would have accrued had there been no early repayment. There are also adjustments for the early return of capital less the repayment of initial expenses that had been borrowed from the treasury. The early repayment fee is given by

Figure 5
Early Repayment Fees



$$G_t = E_0 \frac{L_t}{L_0} - K_t + \frac{1}{P_t} \sum_{j=t+1}^{n} NMI_j (1 + i_H)^{-(j-t)} \tag{18}$$

where *NMI* is defined in equation (14) and does not include any adjustment for early repayments. (In the notation of equation (18), Figure 5 shows G_t/L_t vs. t.)

In Figure 5 the upper line is for the $L_0 = 1,000$, $r_L = 15$ percent combination and the lower line is for $L_0 = 5,000$ and $r_L = 11.6$ percent. Both lines are approximately, but not exactly, linear.

If the results are calculated instead for the situation where capital is used to pay the initial expenses (so the $E_0 L_t / L_0$ term is not needed in the above equation) and a higher interest rate is charged because this method is more expensive, the outcome is almost identical. This indicates that the costs of early repayment are not significantly dependent on the method used to pay for the initial expenses.

We conclude that there are three approaches one can take to model early repayments and assess the risk. Each of the three approaches can be used with the 25 year (full-term) loan model used here or with the seven year (average term) loan model used in Allan et al., (1998).

The first approach takes a best estimate of repayment rates and price to determine the correct average price charged to all borrowers. The problem with this approach is that it is deterministic and the risk of changed early repayment rates only can be assessed by deterministic scenario testing.

The second approach uses an actuarially neutral charging structure so that repayment fees can be charged for early repayments at any time. The fee would leave the internal rate of return of the loan unchanged whatever the time of surrender. Such an approach, which the authors believe will develop further in the U.K., would pass repayment risk to the borrower. It, therefore, would not require a stochastic approach.

The third approach involves a stochastic model of repayments to assess the variability of the internal rate of return given a reasonable model of early repayments. Repayments depend on the repayment fee and the degree of competition in the mortgage market. We believe that such an approach is unnecessary in the U.K. but may be necessary where the market is resistant to actuarially neutral early repayment fees. This is an area we leave for further research.

Our model has assumed a fixed interest rate throughout the term of the loan. In the U.K. most mortgages are variable rate. The results of the model would not be significantly altered if a variable rate were to be used as long as the interest margin (the difference between the loan rate paid by the borrower and the cost of funds for the bank) remained constant. It is the margin rather than the absolute level of interest rates that is important.

A more significant problem would arise when using the model to price fixed rate loans. It would be necessary to deal with the problem of borrowers exercising an option to repay early if variable interest rates fall. An option pricing approach to valuing that option could be used. Otherwise, three approaches are possible. First, the bank could investigate, using deterministic scenario testing, the effect of yield curve changes on profitability. The bank must make appropriate provisions for the exercise of an option to repay early. A second approach is to charge the borrower a penalty for early repayment. This is common in the U.K. The penalties are sufficiently high (for example, six months' interest for early repayment of a five year loan) to provide a significant disincentive for early repayment. Prepayment need not be a problem with the correct charging structure. Third, if the market will not bear prepayment penalties (or regulation prevents them), a fully stochastic model may be appropriate to enable the bank to assess the risk of prepayments caused by yield curve movements as well as other causes.

5.3 Risk Decreasing With Time

In this example the expected default rate and cost of default decrease with the time since the inception of the loan. The effect of varying the interest rate during the course of the loan to reflect this risk is examined.

For mortgages the risk to the bank is concentrated in the first few years of the loan. If a borrower has repayment problems later, the mortgage will be covered by the house value unless housing prices have fallen in nominal terms by more than the amount of the loan repaid. The bank could rearrange the mortgage or take possession of the property.

If risk decreases with time, shouldn't the interest rate? In practice the opposite is often the case as banks try to attract borrowers by providing low introductory rates.

The following example examines the possible benefit of varying the interest rate as the loan progresses. Several parameters are unchanged from previous examples: $r_H = 20$ percent, $r_F = 10$ percent, $r_C = 8$ percent, $r_{T2} = 10$ percent, and $E_t = 0$. Those parameters that differ from previous examples are $L_0 = 100,000$, $E_0 = 500$, $K_0/L_0 = {}^{T2}K_0/L_0 = 2.5$ percent, and $n = 25$ years. Also, default and repayment rates are different, and various values are considered for the loan interest rate. The defaults have the following pattern. The loan loss fraction is given by

$$f_t = \begin{cases} 0.05 + (L_t - L_{36})/L_{36} & t \leq 36 \\ 0 & t > 36. \end{cases}$$

There are no losses after three years as the house value should be enough to cover the mortgage. Before three years the loss fraction decreases with time as part of the loan is repaid, but it is always at least 5 percent of the outstanding loan value. Otherwise, some form of rearrangement may be a more likely outcome than a default. In the first three years the default rate is either 0.2 percent per month or 0.4 percent per month. (Results are given for both values.)

Mortgage loans often end early as persons move before the repayment is complete or transfer to another bank before the full term is finished. The repayment pattern used is

$$R_t = \begin{cases} 0 & t \le 36 \\ 0.015 & t > 36. \end{cases}$$

It is not necessary that the early repayments start at the same time as the defaults stop, but a duration of one to five years is likely to be appropriate for both values. No fee is charged for early terminations of the mortgage in the model considered here. With this repayment pattern, roughly half of mortgages have ended by the seventh year.

How should interest rates be changed to take into account declining default risk? The equation for net monthly income for a set of loans (see, for example, equation (14)) includes

$$NMI = [i_L - (i_L + f)q]L + \text{Terms not involving defaults or } i_L.$$

This suggests that if $i_L - (i_L + f)q$ is kept constant as q and f change, the profitability of a loan will be broadly neutral to default risk. Let $\bar{q}$ and $\overline{(qf)}$ be defined as:

$$\bar{q} = \frac{1}{n}\sum_{t=1}^{n} P_{t-1}q_t$$

$$\overline{(qf)} = \frac{1}{n}\sum_{t=1}^{n} P_{t-1}q_t f_t.$$

Then the invariant requirement leads to an interest rate set by

$$i_L^{Risk} = \frac{(1-\bar{q})i_L^{Mean} + (qf - \overline{(qf)})}{1-q}.$$

When the default rate in the first three years is 0.2 percent per month the weighted averages $\bar{q}$ and $\overline{(qf)}$ are 0.0727 percent and 0.0048 percent, respectively. When the default rate is 0.4 percent these two values are 0.1489 percent and 0.0098 percent, respectively.

The following results are obtained using a simpler formula for the risk interest rate, viz.:

$$i_L^{Risk} = i_L + \frac{q \times f - \overline{(qf)}}{1-q}$$

with i_L the same as the interest rate used in the fixed rate case. This equation is used because the relationship between the mean rate of interest charged under this regime to the rate charged in the fixed rate case is more readily apparent (i.e., they are the same).

With a varying interest rate the loan payment has to be recalculated each month. For month t, let $i_L(t)$ denote the interest rate in effect during month t, i.e., from time $t-1$ to t. Then the payments made at time t, X_t, and L_t are given by

$$\begin{aligned} X_t &= \frac{L_{t-1}}{a_{\overline{n+1-t}|\, i_L(t)}}, \quad \text{and} \\ L_t &= (1 + i_L(t))L_{t-1} - X_t. \end{aligned}$$

The amount received by the bank is $(1 - q_t)P_{t-1}X_t$.

Table 12 shows *NPV* for two default rates and four interest rates that are the smallest, to the nearest 0.01 percent, that give a positive *NPV* at these two default rates, for both the fixed interest rate and declining interest rate case. Two *NPV* values are given for each q and r_L combination, corresponding to these two rate setting methods. In the table r_L means both the rate used in the single rate calculation and the annual equivalent of the rate appearing in the above equation for the risk rate. The mean refers to the straightforward average of the interest rate over the term of the mortgage (not weighted by the survival probability; the weighted average is just r_L).

Table 12

***NPV* for Various q and r_L Combinations**

	Interest Rate				*NPV*	
q	r_L	Mean	Max	Min	F-Rate	V-Rate
0.2%	10.58%	10.54%	10.73%	10.52%	31.7	180.8
0.4%	10.58%	10.49%	10.87%	10.45%	-388.8	-105.4
0.2%	10.71%	10.67%	10.86%	10.65%	455.7	604.1
0.4%	10.71%	10.62%	11.00%	10.58%	16.9	299.0
0.2%	10.53%	10.49%	10.68%	10.47%	-131.3	18.1
0.4%	10.62%	10.53%	10.91%	10.49%	-263.9	19.0

Notes: F-Rate = Fixed interest rate; V-Rate = Variable interest rate.

The net present values are about 150 higher for the variable rate model in the $q = 0.2$ percent cases than in the fixed rate model and

280 higher for the $q = 0.4$ percent cases (for comparison, the initial capital outlay is 2500). Alternatively, instead of making a larger profit at the same (weighted average) interest rate, a lower average rate can be charged, as shown in the last two lines.

Moreover, the importance of the default rate is reduced when this flexible interest rate is used. For example if the rate had been set at 10.58 percent in anticipation of the lower default rate, the reduction in *NPV* that happens if the default rate proves to be 0.4 percent per month is 420 if the fixed rate is used and 286 if the flexible rate is used. This method only works if the risk adjustments to the interest rates are based on the 0.4 percent value. If they are not, the loss is just as great as in the fixed rate case.

This example illustrates that average interest rates can be reduced if risk-related pricing is introduced. The suggested differences in the average rates charged are not large, with reduction in the rates less than 0.1 percent.

6 Summary and Conclusions

Cash flow models that are used in other areas of actuarial work could be used in banking to price loans. Often bank lending is priced by looking at the whole book of business together and ensuring that the margin is adequate to provide the required return on capital on the business as a whole. Adjustments to the margin will generally be made to allow for the risk profiles of different borrowers. A cash flow approach would consider explicitly the cash flows that were expected for a particular category of loans. It would therefore be possible to set an interest margin, appropriate to that category that allowed explicitly for: the capital used to back a particular category of loans; the expenses of a particular category of loans; and the risk of a particular category of loans. The cash flow approach can also handle the many interest rates that are relevant to bank lending. The following interest rates give rise to cash flows: interest charged on the loan; interest paid by the lender to the source of the money lent; hurdle rate of return on equity capital; rate of interest to be paid on debt capital; interest earned on set-aside equity capital.

Having determined the cash flows pertaining to a particular category of loan, the cash flow model can be used to make business decisions such as determining the interest margin to be charged or determining whether the interest margin available on a category of lending in the market makes the loan sufficiently profitable. For marketing reasons, it may be appropriate to give cross subsidies between categories of loans. The cash flow model allows the effect of these cross subsidies to be quantified.

When lending is secured (for example mortgage lending) the effect of defaults is not significant but in general defaults are important. The model has been extended to deal with defaults but more empirical work is necessary to find reasonable models of default rates and the loss incurred by a bank on defaults. Various ways of dealing with prepayments are discussed. Prepayments can cause difficulty for two reasons: first initial expenses may not be recouped if a loan is prepaid; second, if a loan is given at a fixed rate of interest there is a financial interest rate option against the lender. In the U.K. market, it may be possible to develop charging structures so that early repayments do not have a material financial effect on the bank (prepayment penalties are common). Where this is not possible, stochastic modeling of prepayments should be performed.

The model is extended to include pricing for loan products which involve cash backs and prepayment fees. The sensitivity of the model to various parameters is tested and it is found that expenses; the interest margin; the size of loan; and the duration of the loan are the most important parameters. Banks may wish to differentially price loans to a greater extent than is currently the case to allow for size and duration of a loan. Alternatively, as has been mentioned above, prepayment fees or other charging structures could be used to ensure that loans which are prepaid are still profitable.

The main areas for further work are the development of better models for estimating the costs of default and the modeling of prepayments where charging structures do not make the bank indifferent to prepayments.

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Appendix A: How Credit Risk is Assessed

Corporate Loans

A bank can group large borrowers according to expected default risk. The allocation of a borrower to a risk group generally is based on accounting ratios. Statistical techniques (regression or multiple discriminant analysis) are applied to historical data on bankruptcies or loan defaults with these accounting ratios to produce a set of weights for the ratios. Any potential borrower's accounts can be studied to provide a score that is intended to be a predictor of default risk.

The procedure has been in widespread use since the 1960s, and some of the formulae used have been published. It is often found, however, that weighting factors determined from one data set are not the same as those from another set: e.g., U.S.-derived weights are not applicable in the U.K., and 1970s values are not useful now.

A review of many analyses of corporate default is presented in Altman (1983). For a paper relating to risk factors in the U.K., see Taffler (1982). References to more recent papers are found in Altman (1996). Altman (1996) also provides a formula for a score that is a predictor of default risk, with a higher score indicating a lower probability of default:

$$Score = c_0 + c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4$$

where

$$\begin{aligned} X_1 &= \text{Working capital/total assets;} \\ X_2 &= \text{Retained earnings/total assets;} \\ X_3 &= \text{Earnings before interest and taxes/total assets; and} \\ X_4 &= \text{Equity (book value)/total liabilities.} \end{aligned}$$

Also, c_1, c_2, c_3, and c_4 are positive constants.

Personal Loans: Credit Scoring

The method of assessing the risks in personal loans is called *credit scoring*. The statistical techniques used are similar to those used for large corporate loans (e.g., discriminant analysis), but the factors in the model change (there are no audited accounts to use).

There is considerable danger in using past data to predict future bad debts. The economic background is likely to influence the overall level of bad debts. Within this overall trend, the credit score should indicate some ranking of risk.

For a given type of loan, the most important data are provided by credit agencies. Evidence of existing bad debts with other banks and evidence of successful maintenance of credit repayments are relevant. Other demographic data as provided on loan application forms may be used, but are not always major influences. (One reason is that banks find that short application forms are useful in attracting customers.) A third source of information is behavior, i.e., the credit history of a loan applicant who is an existing customer of the organization. All of the factors can be combined onto a scorecard, and historical data will provide default rates and costs of default versus score.

The construction of a scorecard usually is done by a specialist agency. Typically a new scorecard will be prepared every two or three years. Around 1,000 to 2,000 bad loans are needed (and an equal number of good loans) to provide statistically sound weights. Given that a default rate of only a few percent a year is not uncommon, this number of bad loans requires a large portfolio, a long base period, or a weak definition of bad. In practice the last option is likely to be selected.

It is appropriate to have different scorecards for different products (e.g., mortgages and credit cards) and for different categories of customer (e.g., new or existing customer).

There is an important limitation on the reliability of default rate predictions; the data are for a select set of the population, i.e., persons accepted for loans a few years ago. Risk factors that are unimportant among this group may be significant in the population of future loan applicants.

Once a score has been calculated for a loan applicant, the most common approach in bank lending is a straight accept or reject decision (e.g., accept the application if the score is greater than, say, 100). This contrasts with risk-based pricing in insurance where premium rates vary with risk rather than being one rate for all accepted. The equivalent response in terms of lending would be to charge a rate of interest that varies with risk. This method has been introduced in some areas of bank lending. An alternative method, used in practice with credit cards, is for a bank to operate several cards (perhaps under different names) with different interest rates. To be accepted for a low interest rate card, a higher score will be needed than for the higher interest rate cards.

Appendix B: Uniform Repayments to the Treasury

This section expands possible ways in which the bank could repay money to the bank's treasury.

The paper has been based on the assumption that the bank always owes to the treasury an amount equal to the amount that the borrower owes. (Initial expenses are also paid in the same pattern.) An alternative is that the initial amount borrowed from the treasury (for loans and initial expenses) are amortized over the period of the loans and paid in equal installments irrespective of whether some loans default or are repaid early.

If there is a difference between the amounts owed by the bank (to the treasury) and to the bank (by the borrower) this does not mean that there is idle cash available. The bank holds no money other than capital (which is invested in the wholesale cash market); all of the money it receives is immediately paid to the treasury or assigned to the providers of capital as profit.

Under the uniform repayment scheme the equation for net monthly income, which is comparable to the equation (14), is:

$$\begin{aligned} NMI_t &= XP_t^{(qr)}(1-q_t) - \frac{L_0+E_0}{a_{\overline{n}|\,i_F}} \\ &\quad + i_C P_{t-1}^{(qr)} K_{t-1} + (P_{t-1}^{(qr)} K_{t-1} - P_t^{(qr)} K_t) - P_{t-1}^{(qr)D} K_{t-1}(i_D - i_C) \\ &\quad - E_t + q_t P_{t-1}^{(qr)}(1-f_t)L_{t-1} + R_t(1-q_t)P_{t-1}^{(qr)}(L_t + G_t). \end{aligned}$$

(Everything is the same except for payments to the treasury.)

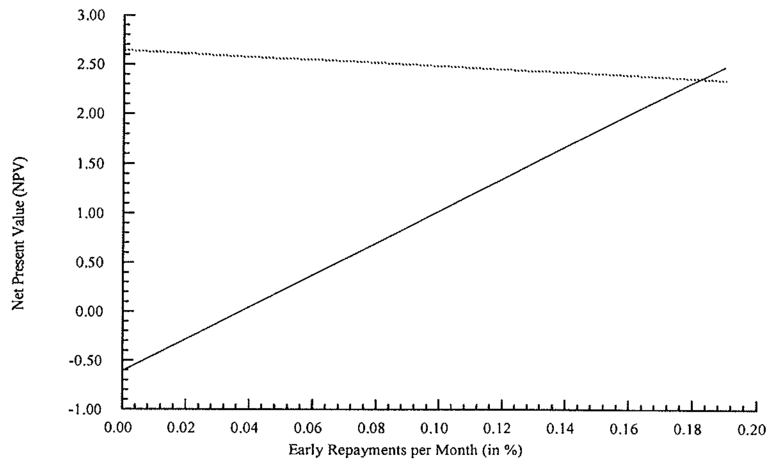
Figure B1 illustrates the significance of the two fund repayment patterns. The parameters used in producing this figure are the same as for the second spreadsheet example, except that early repayments are included (but there are no early repayments allowed in the first year). A loan interest rate of $r_L = 11.45$ percent is chosen to make *NPV* close to zero. (At the maximum early closure rate shown, 0.2 percent, 95 percent of loans last the duration.)

Figure B1 shows that even without premature repayments there is a difference in the value of this loan under the two methods of funding. The proportional method gives a higher value because there is a slight delay in the timing of repayments to the treasury. (These repayments are initially less than in the uniform case, but are greater later. In total they are a little larger under the proportional method.)

Also, early repayments reduce the value of the loan, as far as the bank is concerned, when the proportional method of fund repayment is used. Early repayments are welcome under the uniform scheme, however. Although it may seem unreasonable for a bank to want a profitable loan to stop, the improvement shown in the figure is not wrong. The result depends on the bank being able to secure funds from the treasury at a rate of interest below the hurdle rate that can be retained after some of the loans have been repaid (or defaulted).

The figure illustrates the point that the way the funding is arranged or the method of bookkeeping selected can be more important in de-

Figure B1
Different Repayment Patterns for Funds



termining the profitability of a class of loans than a key feature of the loan (in this case the early repayment pattern).

The steepness of the line for the uniform repayment case shows that the profitability of the loan is likely to be more volatile if this scheme is used. Also, the internal rate of return of loans financed by the uniform repayment scheme is more sensitive to the default rates, when the default rates are high than they would be under the proportional repayment scheme. In this sense there is more risk accepted by the providers of capital when a uniform repayment plan is used, and a higher return therefore may be required.